

ASYMPTOTIC BEHAVIOR AND REPRESENTATION OF SOLUTIONS TO A VOLTERRA KIND OF EQUATION WITH A SINGULAR KERNEL

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ABSTRACT. Let A be a densely defined closed, linear ω -sectorial operator of angle $\theta \in [0, \frac{\pi}{2})$ on a Banach space X for some $\omega \in \mathbb{R}$. We give an explicit representation (in terms of some special functions) and study the precise asymptotic behavior as time goes to infinity of solutions to the following diffusion equation with memory: $u'(t) = Au(t) + (\kappa * Au)(t)$, $t > 0$, $u(0) = u_0$, associated with the (possible) singular kernel $\kappa(t) = \alpha e^{-\beta t} \frac{t^{\mu-1}}{\Gamma(\mu)}$, $t > 0$, where $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\beta \geq 0$ and $0 < \mu \leq 1$.

1. INTRODUCTION

In the present paper, we consider the following Volterra kind of system

$$\begin{cases} u'(t) = Au(t) + \int_0^t \kappa(t-s)Au(s)ds, & t > 0, \\ u(0) = u_0, \end{cases} \quad (1.1)$$

where A with domain $D(A)$ is a densely defined linear sectorial operator on a Banach space X , u_0 is a given function in X and the possible (singular) kernel κ is given by

$$\kappa(t) = \alpha e^{-\beta t} \frac{t^{\mu-1}}{\Gamma(\mu)}, \quad t > 0, \quad (1.2)$$

with $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\beta \geq 0$ and $0 < \mu \leq 1$. We mention that the kernel $\kappa \in L^1_{\text{loc}}([0, \infty))$, and is even in $L^1((0, \infty))$ if $\beta > 0$, but it is not in $W^{1,1}_{\text{loc}}([0, \infty))$ if $0 < \mu < 1$. It is straightforward to verify that the system (1.1) is equivalent to the integral equation

$$u(t) = u_0 + \int_0^t (1 + 1 * \kappa)(t-s)Au(s)ds, \quad t \geq 0. \quad (1.3)$$

By [15, Chapter I], the well-posedness of the system (1.1) (or equivalently, the integral equation (1.3)) is equivalent to the existence of a family of bounded linear operators $(S^\mu_{\alpha,\beta}(t))_{t \geq 0}$ on X , which we shall call **(α, β, μ) -resolvent family**, verifying the following properties.

- $S^\mu_{\alpha,\beta}(0) = I$ and $S^\mu_{\alpha,\beta}(\cdot)$ is strongly continuous on $[0, \infty)$.
- $S^\mu_{\alpha,\beta}(t)x \in D(A)$ and $S^\mu_{\alpha,\beta}(t)Ax = AS^\mu_{\alpha,\beta}(t)x$ for all $x \in D(A)$ and $t \geq 0$.
- For all $x \in D(A)$ and $t \geq 0$, the resolvent equation holds:

$$S^\mu_{\alpha,\beta}(t)x = x + \int_0^t (1 + 1 * k)(t-s)AS^\mu_{\alpha,\beta}(s)x ds. \quad (1.4)$$

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In that case, for every $u_0 \in X$, the unique (mild) solution of (1.1) is given by

$$u(t) = S_{\alpha,\beta}^\mu(t)u_0, \quad t \geq 0. \quad (1.5)$$

We give the idea how to get (1.5) without further detail. In fact, uniqueness of solutions is easy to establish. Taking the Laplace transform of both sides of the first equation in (1.1) we get that

$$\lambda \widehat{u}(\lambda) - u(0) = A\widehat{u}(\lambda) + \frac{\alpha}{(\lambda + \beta)^\mu} A\widehat{u}(\lambda).$$

Thus

$$\begin{aligned} & \frac{(\lambda + \beta)^\mu + \alpha}{(\lambda + \beta)^\mu} \left(\frac{\lambda(\lambda + \beta)^\mu}{(\lambda + \beta)^\mu + \alpha} - A \right) \widehat{u}(\lambda) \\ &= [g(\lambda)]^{-1} (h(\lambda) - A) \widehat{u}(\lambda) = u_0, \end{aligned} \quad (1.6)$$

where we have set

$$g(\lambda) := \frac{(\lambda + \beta)^\mu}{(\lambda + \beta)^\mu + \alpha} \quad \text{and} \quad h(\lambda) := \frac{\lambda(\lambda + \beta)^\mu}{(\lambda + \beta)^\mu + \alpha} = \lambda g(\lambda).$$

Note that $g(\lambda)$ and $h(\lambda)$ are well-defined for all λ with $\operatorname{Re}(\lambda) > 0$. Taking also the Laplace transform of both sides of (1.4) with $x = u_0$ we get that

$$\widehat{S}_{\alpha,\beta}^\mu(\lambda)u_0 = \frac{1}{\lambda}u_0 + \left(\frac{1}{\lambda} + \frac{\widehat{k}(\lambda)}{\lambda} \right) A\widehat{S}_{\alpha,\beta}^\mu(\lambda)u_0.$$

Calculating, we obtain that

$$\begin{aligned} & \frac{(\lambda + \beta)^\mu + \alpha}{(\lambda + \beta)^\mu} \left(\frac{\lambda(\lambda + \beta)^\mu}{(\lambda + \beta)^\mu + \alpha} - A \right) \widehat{S}_{\alpha,\beta}^\mu(\lambda)u_0 \\ &= [g(\lambda)]^{-1} (h(\lambda) - A) \widehat{S}_{\alpha,\beta}^\mu(\lambda)u_0 = u_0. \end{aligned} \quad (1.7)$$

It follows from (1.6) and (1.7) that if $h(\lambda) \in \rho(A)$, then

$$\widehat{u}(\lambda) = \widehat{S}_{\alpha,\beta}^\mu(\lambda)u_0 = g(\lambda) (h(\lambda) - A)^{-1} u_0. \quad (1.8)$$

Now taking the inverse Laplace transform of (1.8) we get that the unique (mild) solution of (1.1) is given by (1.5). For more details on this topic we refer to the monograph [15].

Next, we assume that the system (1.1) is well-posed and we denote by $S_{\alpha,\beta}^\mu$ the associated (α, β, μ) -resolvent family.

Definition 1.1. *We shall say that $S_{\alpha,\beta}^\mu$ is **exponentially bounded, or of type** $(M, \tilde{\omega})$, if there exist two constants $M > 0$ and $\tilde{\omega} \in \mathbb{R}$ such that*

$$\|S_{\alpha,\beta}^\mu(t)\| \leq M e^{\tilde{\omega}t}, \quad \forall t \geq 0. \quad (1.9)$$

*The resolvent family $S_{\alpha,\beta}^\mu$ will be said to be **uniformly exponentially stable** if (1.9) holds with some constants $M > 0$ and $\tilde{\omega} < 0$.*

As we have mentioned above, the well-posedness of the system (1.1) and many other properties of resolvent families have been intensively studied in the monograph [15]. Similar problems have been also considered in [2, 3, 9, 11, 13] and their references. The asymptotic behavior as time goes to ∞ of solutions to some similar integro-differential equations in finite dimension involving kernels as the one in (1.2) have been also studied in [16].

The main concerns in the present paper are the following.

- (1) Study the precise asymptotic behavior as time goes to infinity of solutions to the system (1.1). That is, we would like to characterize the minimum real number $\tilde{\omega}$ in the estimate (1.9). Of particular interest will be to find some precise conditions on the parameters α , β , μ and the operator A that shall imply the uniform exponential stability of the family $S_{\alpha,\beta}^\mu$, and hence, of solutions to the system (1.1).
- (2) Find an explicit representation of solutions to the system (1.1) in terms of some special functions. The Mittag-Leffler functions and its generalizations will be the natural candidate.

We mention that if $\mu = 1$ and $\beta > 0$ in (1.2), that is, $\kappa(t) = \alpha e^{-\beta t}$, then the asymptotic behavior of the associated resolvent family has been completely studied in [1] by using some semigroups method. See also [6, Chapter VI] where the well-posedness of the system (1.1) for a general kernel $\kappa \in W^{1,1}((0, \infty))$ has been obtained by using again the method of semigroups. One of our concerns is to extend the results contained in [1] to the case $0 < \mu < 1$, where the method of semigroups cannot be used. Note that in our situation $\kappa \notin W_{\text{loc}}^{1,1}([0, \infty))$. If $\beta = 0$ and $0 < \mu < 1$, then it is well-known (see e.g. [4, 5, 7, 10, 14] and their references) that the corresponding $(\alpha, 0, \mu)$ -resolvent family is never uniformly exponentially stable. In this case, solutions may decay but only at most polynomial. We shall see here that the situation is different if $\beta > 0$. More precisely, assuming that the operator A is ω -sectorial of angle θ (see Section 2 below for the definition) for some $\omega < 0$, $0 \leq \theta \leq \tilde{\mu}\frac{\pi}{2}$, where $0 < \mu < \tilde{\mu} \leq 1$, and that $\beta + \omega \leq 0$, we obtain the following result.

- (a) If $\alpha > 0$, then the family is uniformly exponentially stable with $\tilde{\omega} = -\beta$.
- (b) If $\alpha < 0$ and $\alpha + \beta^\mu \geq |\alpha|$, then the family is exponentially bounded with exponential bound $\tilde{\omega} = -\left(\beta - (\alpha\omega)^{\frac{1}{\mu+1}}\right)$. This will imply the uniform exponential stability of the family if in addition $\beta^{\mu+1} > \alpha\omega$.

The rest of the paper is structured as follows. In Section 2 we state the main results where some generation results of (α, β, μ) -resolvent families and the precise asymptotic behavior as time goes to infinity of the resolvent family have been obtained. Section 3 contains the proof of the generation theorem. In Section 4 we give the proof of the asymptotic behavior as time goes to infinity of the resolvent family. Finally in Section 5, assuming that $A = \rho I$ for some $\rho \in \mathbb{R}$ or A is a self-adjoint operator on $L^2(\Omega)$ (where $\Omega \subset \mathbb{R}^N$ is a bounded open set) with compact resolvent, we obtain an explicit representation of solutions to the system (1.1) (and hence of the associated resolvent family) in terms of Mittag-Leffler type of functions.

2. MAIN RESULTS

Let X be a Banach space with norm $\|\cdot\|$. For a closed, linear and densely defined operator A on X , we denote by $\sigma(A)$ and $\rho(A)$ the spectrum and the resolvent set of A , respectively. The operator A is said to be **ω -sectorial of angle θ** if there exist $\theta \in [0, \pi/2)$ and $\omega \in \mathbb{R}$ such that

$$w + \Sigma_\theta := \left\{ \omega + \lambda, \lambda \in \mathbb{C} : |\arg(\lambda)| < \theta + \frac{\pi}{2} \right\} \subset \rho(A),$$

and one has the estimate

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - \omega|}, \quad \forall \lambda \in \{\omega + \Sigma_\theta\} \setminus \{\omega\}.$$

If $\omega = 0$, we shall only say that A is **sectorial of angle θ** . More details on sectorial operators can be found in [6, 8] and they references.

In this section we state the main results of the paper. We start with the generation theorem.

Theorem 2.1. *Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\beta \geq 0$ and $0 < \mu \leq 1$. Assume any one of the following two conditions.*

- (a) $\alpha > 0$ and A is a sectorial operator of angle $\mu\frac{\pi}{2}$.
- (b) $\alpha < 0$, $\alpha + \beta^\mu \geq |\alpha|$ and A is a sectorial operator of angle $\mu\frac{\pi}{2}$.

Then A generates an (α, β, μ) -resolvent family $(S_{\alpha, \beta}^\mu(t))_{t \geq 0}$ of type $(M, \tilde{\omega})$ for every $\tilde{\omega} > 0$.

Corollary 2.2. Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\beta \geq 0$, $0 < \mu \leq 1$ and $\omega \in \mathbb{R}$. Assume any one of the following two conditions.

- (a) $\alpha > 0$ and A is an ω -sectorial operator of angle $\mu\frac{\pi}{2}$.
- (b) $\alpha < 0$, $\alpha + \beta^\mu \geq |\alpha|$ and A is an ω -sectorial operator of angle $\mu\frac{\pi}{2}$.

Then A generates an (α, β, μ) -resolvent family $(S_{\alpha, \beta}^\mu(t))_{t \geq 0}$ of type $(M, \tilde{\omega})$ for some $\tilde{\omega} > 0$.

As we have mentioned above, the asymptotic behavior as time goes to infinity of the resolvent family for the case $\mu = 1$ has been completely studied in [1]. Therefore we concentrate on the case $0 < \mu < 1$. The following estimates of the resolvent family is the second main result of the paper.

Theorem 2.3. Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\beta \geq 0$, $0 < \mu < \tilde{\mu} \leq 1$ and $\omega < 0$. Assume that $\beta + \omega \leq 0$. Then the following assertions hold.

- (a) If $\alpha > 0$ and A is ω -sectorial of angle $\tilde{\mu}\frac{\pi}{2}$, then there exists a constant $C > 0$ such that the resolvent family $(S_{\alpha, \beta}^\mu(t))_{t \geq 0}$ generated by A satisfies the estimate

$$\|S_{\alpha, \beta}^\mu(t)\| \leq Ce^{-\beta t}, \quad \forall t \geq 0. \quad (2.1)$$

- (b) If $\alpha < 0$, $\alpha + \beta^\mu \geq |\alpha|$ and A is ω -sectorial of angle $\tilde{\mu}\frac{\pi}{2}$, then there exists a constant $C > 0$ such that the resolvent family $(S_{\alpha, \beta}^\mu(t))_{t \geq 0}$ generated by A satisfies the estimate

$$\|S_{\alpha, \beta}^\mu(t)\| \leq C(1 + \alpha\omega t^{\mu+1})e^{-(\beta - (\alpha\omega)^{\frac{1}{\mu+1}})t}, \quad \forall t \geq 0. \quad (2.2)$$

3. PROOF OF THEOREM 2.1 AND COROLLARY 2.2

Before given the proof of the generation theorem we need some intermediate results. First, we recall the following fundamental result from [15, Chapter I, Theorem 1.3].

Theorem 3.1. Let $\mathbf{a} \in L_{\text{loc}}^1([0, \infty))$ satisfy

$$\int_0^\infty e^{-\tilde{\omega}t} |\mathbf{a}(t)| dt < \infty. \quad (3.1)$$

Let A with domain $D(A)$ be a linear operator on X with dense domain. Then A generates a resolvent family (associated with \mathbf{a}) of type $(M, \tilde{\omega})$ on X if and only if the following two conditions hold.

- (a) $\hat{\mathbf{a}}(\lambda) \neq 0$ and $\frac{1}{\hat{\mathbf{a}}(\lambda)} \in \rho(A)$ for all $\lambda > \tilde{\omega}$.
- (b) The mapping $\lambda \mapsto H(\lambda) := \frac{1}{\lambda} (I - \hat{\mathbf{a}}(\lambda)A)^{-1}$ satisfies the estimate

$$\|H^{(n)}(\lambda)\| \leq \frac{Mn!}{(\lambda - \tilde{\omega})^{n+1}}, \quad \lambda > \tilde{\omega}, \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

where $H^{(n)}(\lambda) = \frac{d^n H}{d\lambda^n}(\lambda)$.

The following result will be also useful.

Lemma 3.2. Let $\alpha \in \mathbb{R}$, $\beta \geq 0$, $0 < \mu \leq 1$ and define $g(\lambda) := \frac{(\lambda + \beta)^\mu}{(\lambda + \beta)^\mu + \alpha}$ for all λ with $\text{Re}(\lambda) > 0$. Then the following assertions hold.

- (a) If $\alpha \geq 0$, then $|g(\lambda)| \leq 1$.
- (b) If $\alpha < 0$ and $\alpha + \beta^\mu \geq |\alpha|$, then $|g(\lambda)| \leq \frac{\beta^\mu}{\alpha + \beta^\mu} \leq \frac{\beta^\mu}{|\alpha|}$.

Proof. Let $\alpha \in \mathbb{R}$, $\beta \geq 0$ and $0 < \mu \leq 1$.

(a) This assertion is obvious.

(b) Now assume that $\alpha < 0$ and $\beta^\mu + \alpha \geq |\alpha|$. First, observe that

$$|g(\lambda)| = \left| 1 - \frac{\alpha}{(\lambda + \beta)^\mu + \alpha} \right| \leq 1 + \frac{|\alpha|}{|(\lambda + \beta)^\mu - \beta^\mu + (\alpha + \beta^\mu)|}. \quad (3.2)$$

We claim that

$$\operatorname{Re}[(\lambda + \beta)^\mu - \beta^\mu] > 0 \text{ for all } \operatorname{Re}(\lambda) > 0. \quad (3.3)$$

We show first that

$$\operatorname{Re}[(\lambda + \beta)^\mu] > (\operatorname{Re}(\lambda) + \beta)^\mu \text{ for all } \operatorname{Re}(\lambda) > 0. \quad (3.4)$$

Indeed, let $z := \lambda + \beta$. We have to show that $\operatorname{Re}(z^\mu) > (\operatorname{Re}(z))^\mu$, that is, $|z|^\mu \cos(\theta\mu) > (|z| \cos(\theta))^\mu$, where $z = |z| \cos(\theta)$ with $|\theta| < \frac{\pi}{2}$. Let $f(\mu) := \cos(\mu\theta) - \cos^\mu(\theta)$, $0 \leq \mu \leq 1$. Then

$$f''(\mu) = -\theta^2 \cos(\mu\theta) - (\ln(\cos(\theta)))^2 \cos^\mu(\theta) < 0 \text{ for all } 0 < \mu \leq 1,$$

and this implies that f is a concave function. Since $f(0) = f(1) = 0$, we have that the graph of f is above the straight line joining the points $(0, f(0)) = (0, 0)$ and $(1, f(1)) = (1, 0)$. This implies that $f(\mu) \geq 0$ for all $0 \leq \mu \leq 1$. Thus $\cos(\mu\theta) \geq \cos^\mu(\theta)$ for all $0 \leq \mu \leq 1$ and we have shown (3.4). The claim (3.3) follows from (3.4).

On the other hand, since by assumption $\alpha + \beta^\mu \geq |\alpha| > 0$, it follows from (3.3) that

$$|\operatorname{Re}[(\lambda + \beta)^\mu - \beta^\mu + (\alpha + \beta^\mu)]| > \alpha + \beta^\mu. \quad (3.5)$$

Using (3.5), we get from (3.2) that

$$|g(\lambda)| \leq 1 + \frac{|\alpha|}{\alpha + \beta^\mu} = \frac{\beta^\mu}{\alpha + \beta^\mu} \leq \frac{\beta^\mu}{|\alpha|},$$

and the proof is finished. \square

We also need the following result.

Lemma 3.3. *Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\beta \geq 0$, $0 < \mu \leq 1$ and define the function*

$$h(\lambda) = \frac{\lambda(\lambda + \beta)^\mu}{(\lambda + \beta)^\mu + \alpha}, \quad \lambda \in \mathbb{C}, \operatorname{Re}(\lambda) > 0. \quad (3.6)$$

Assume any one of the following two conditions.

(a) $\alpha > 0$.

(b) $\alpha < 0$ and $\alpha + \beta^\mu \geq |\alpha|$.

Then

$$|\arg(h(\lambda))| \leq (1 + \mu)|\arg(\lambda)|. \quad (3.7)$$

Proof. Let h be given by (3.6) where $\lambda = re^{i\theta}$ with $|\theta| < \frac{\pi}{2}$ and $r > 0$. Without any restriction we may assume that $\theta \geq 0$. Then

$$\begin{aligned} \arg(h(re^{i\theta})) &= \operatorname{Im}(\ln(h(re^{i\theta}))) = \operatorname{Im} \int_0^\theta \frac{d}{dt} \ln(h(re^{it})) dt \\ &= \operatorname{Im} \int_0^\theta \frac{h'(re^{it}) i re^{it}}{h(re^{it})} dt. \end{aligned} \quad (3.8)$$

Moreover a simple calculation gives

$$\lambda \frac{h'(\lambda)}{h(\lambda)} = 1 + \left(\frac{\alpha\mu}{(\lambda + \beta)^\mu + \alpha} \right) \left(\frac{\lambda}{\lambda + \beta} \right).$$

(a) Assume that $\alpha > 0$. Then

$$\left| \lambda \frac{h'(\lambda)}{h(\lambda)} \right| \leq 1 + \mu \left| \frac{\alpha}{(\lambda + \beta)^\mu + \alpha} \right| \left| \frac{\lambda}{\lambda + \beta} \right| \leq 1 + \mu. \quad (3.9)$$

Using (3.9) we get from (3.8) that

$$\left| \operatorname{Im} \int_0^\theta \frac{h'(re^{it})ire^{it}}{h(re^{it})} dt \right| \leq \int_0^\theta \left| \frac{h'(re^{it})ire^{it}}{h(re^{it})} \right| dt \leq \int_0^\theta (1 + \mu) dt \leq (1 + \mu)\theta,$$

and we have shown (3.7).

(b) Now assume that $\alpha < 0$ and $\beta^\mu + \alpha \geq |\alpha|$. Then using (3.5) we get that

$$\left| \lambda \frac{h'(\lambda)}{h(\lambda)} \right| \leq 1 + \mu \left| \frac{\alpha}{(\lambda + \beta)^\mu + \alpha} \right| \left| \frac{\lambda}{\lambda + \beta} \right| \leq 1 + \frac{\mu|\alpha|}{\alpha + \beta^\mu} \leq 1 + \mu. \quad (3.10)$$

It follows from (3.8) and (3.10) that

$$\begin{aligned} \left| \operatorname{Im} \int_0^\theta \frac{h'(re^{it})ire^{it}}{h(re^{it})} dt \right| &\leq \int_0^\theta \left| \frac{h'(re^{it})ire^{it}}{h(re^{it})} \right| dt \\ &\leq \int_0^\theta 1 + \frac{\mu|\alpha|}{\alpha + \beta^\mu} dt \leq (1 + \mu)\theta. \end{aligned}$$

We have shown (3.7) and the proof is finished. \square

Proof of Theorem 2.1. First we note that compare with Theorem 3.1 we have that

$$\mathbf{a}(t) = (1 + 1 * \kappa)(t), \quad t > 0,$$

so that its Laplace transform is given by

$$\widehat{\mathbf{a}}(\lambda) = \frac{1}{\lambda} + \frac{\widehat{\kappa}(\lambda)}{\lambda} = \frac{(\lambda + \beta)^\mu + \alpha}{\lambda(\lambda + \beta)^\mu}.$$

It is clear that for every $\tilde{\omega} > 0$ we have that

$$\int_0^\infty e^{-\tilde{\omega}t} |\mathbf{a}(t)| dt = \int_0^\infty e^{-\tilde{\omega}t} |(1 + 1 * \kappa)(t)| dt < \infty.$$

Hence, we have to show that the two conditions in Theorem 3.1 are satisfied. It is easy to see that under the assumptions (a) or (b) we have that $\widehat{\mathbf{a}}(\lambda) \neq 0$ for all λ with $\operatorname{Re}(\lambda) > 0$.

Next, we claim that

$$\frac{1}{\widehat{\mathbf{a}}(\lambda)} = h(\lambda) = \frac{\lambda(\lambda + \beta)^\mu}{(\lambda + \beta)^\mu + \alpha} \in \rho(A) \quad \text{for all } \lambda \text{ with } \operatorname{Re}(\lambda) > 0. \quad (3.11)$$

It follows from Lemma 3.3 that in both cases (a) and (b), we have that

$$h(\lambda) \in \Sigma_{\frac{\pi\mu}{2}} \quad \text{for all } \lambda \text{ with } \operatorname{Re}(\lambda) > 0.$$

This implies that the function $H(\lambda) = g(\lambda)(h(\lambda) - A)^{-1}$ is well defined, where $g(\lambda)$ is given by

$$g(\lambda) = \frac{(\lambda + \beta)^\mu}{(\lambda + \beta)^\mu + \alpha} = \frac{h(\lambda)}{\lambda}.$$

Since A is a sectorial operator of angle $\mu\frac{\pi}{2}$ (in both cases (a) and (b)), we have that there exists a constant $M > 0$ such that for all λ with $\operatorname{Re}(\lambda) > 0$,

$$\|\lambda H(\lambda)\| = |h(\lambda)| \|(h(\lambda) - A)^{-1}\| \leq M \frac{|h(\lambda)|}{|h(\lambda)|} = M. \quad (3.12)$$

Moreover a simple calculation gives that in both cases,

$$\begin{aligned} \lambda^2 H'(\lambda) &= \mu \frac{\lambda}{\lambda+\beta} \lambda H(\lambda) - \mu \frac{\lambda}{\lambda+\beta} g(\lambda) \lambda H(\lambda) - \lambda^2 H(\lambda)^2 \\ &\quad - \mu \frac{\lambda}{\lambda+\beta} \lambda^2 H(\lambda)^2 + g(\lambda) \lambda^2 H(\lambda)^2 \mu \frac{\lambda}{\lambda+\beta}. \end{aligned}$$

Note that in the case (b) we have that $\alpha + \beta^\mu \geq (1 + \mu)\alpha + \beta^\mu \geq |\alpha| + \mu\alpha > 0$. Since the function g is bounded (by Lemma 3.2), then using (3.12), we get that there exists a constant $M_1 > 0$ such that

$$\|\lambda^2 H'(\lambda)\| \leq M_1 \quad \text{for all } \lambda \text{ with } \operatorname{Re}(\lambda) > 0. \quad (3.13)$$

Combining (3.12) and (3.13) we get that there exists a constant $M > 0$ such that

$$\|\lambda H(\lambda) + \lambda^2 H'(\lambda)\| \leq M \quad \text{for all } \lambda \text{ with } \operatorname{Re}(\lambda) > 0. \quad (3.14)$$

By [15, Proposition 0.1], the estimate (3.14) implies that

$$\|H^{(n)}(\lambda)\| \leq \frac{Mn!}{\lambda^{n+1}}, \quad \forall \lambda > 0, \quad n \in \mathbb{N}_0. \quad (3.15)$$

From (3.15) we also have that for every $\tilde{\omega} > 0$,

$$\|H^{(n)}(\lambda)\| \leq \frac{Mn!}{(\lambda - \tilde{\omega})^{n+1}}, \quad \forall \lambda > \tilde{\omega}, \quad n \in \mathbb{N}_0.$$

Finally, it follows from Theorem 3.1 that A generates an (α, β, μ) -resolvent family $(S_{\alpha, \beta}^\mu(t))_{t \geq 0}$ of type $(M, \tilde{\omega})$ and the proof is finished. \square

Proof of Corollary 2.2. Assume that $\alpha > 0$, or $\alpha < 0$ and $\alpha + \beta^\mu \geq |\alpha|$ and that A is ω -sectorial of angle $\mu\frac{\pi}{2}$. The claim follows from the decomposition $A = (\omega I + A) - \omega I$, using Theorem 2.1 and the perturbation result of resolvent families contained in [12, Corollary 3.2]. \square

We make some comments about the results obtained in Theorem 2.1 and Corollary 2.2.

Remark 3.4. We notice that even if we assume that the operator A is ω -sectorial for some $\omega < 0$, then uniform exponential stability of the family $(S_{\alpha, \beta}^\mu(t))_{t \geq 0}$ cannot be derived from Theorem 2.1 and Corollary 2.2. In fact, the integral condition (3.1) on the kernel $\mathbf{a}(t) = (1 + 1 * \kappa)(t)$ is only satisfied for $\tilde{\omega} > 0$. Therefore some new ideas are needed in the study of the uniform exponential stability of the family which is one of the main goals of the present paper.

4. PROOF OF THEOREMS 2.3

In this section we give the proof of Theorems 2.3. For this we need the following result.

Lemma 4.1. Let $\beta \geq 0$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $0 < \mu \leq 1$ and define the function

$$\tilde{h}(\lambda) := \frac{(\lambda - \beta)\lambda^\mu}{\lambda^\mu + \alpha}. \quad (4.1)$$

If $\operatorname{Re}(\lambda) < 0$ and $|\lambda^\mu| \geq 2|\alpha|$, then

$$|\arg(\tilde{h}(\lambda))| \leq (1 + \mu)|\arg(\lambda)|. \quad (4.2)$$

Proof. We proceed as in the proof of Lemma 3.3. Let \tilde{h} be given by (4.1) with $\lambda = re^{i\theta}$ where $|\theta| > \frac{\pi}{2}$ and $r^\mu \geq 2|\alpha|$. Without any restriction we may assume that $\theta > \frac{\pi}{2}$. A simple calculation gives that

$$\lambda \frac{\tilde{h}'(\lambda)}{\tilde{h}(\lambda)} = \frac{\lambda}{\lambda - \beta} + \mu \frac{\alpha}{\lambda^\mu + \alpha}. \quad (4.3)$$

Recall that

$$\arg(\tilde{h}(re^{i\theta})) = \text{Im}(\ln(\tilde{h}(re^{i\theta}))) = \text{Im} \int_0^\theta \frac{\tilde{h}'(re^{it})ire^{it}}{\tilde{h}(re^{it})} dt.$$

Since $\text{Re}(\lambda) < 0$ and $|\lambda^\mu| \geq 2|\alpha|$, we have that $|\lambda^\mu + \alpha| \geq ||\lambda^\mu| - |\alpha|| \geq |\alpha|$. Thus it follows from (4.3) that

$$\left| \lambda \frac{\tilde{h}'(\lambda)}{\tilde{h}(\lambda)} \right| \leq 1 + \mu.$$

Therefore

$$\left| \text{Im} \int_0^\theta \frac{h'(re^{it})ire^{it}}{h(re^{it})} dt \right| \leq \int_0^\theta \left| \frac{h'(re^{it})ire^{it}}{h(re^{it})} \right| dt \leq \int_0^\theta (1 + \mu) dt \leq (1 + \mu)\theta,$$

and the proof is finished. \square

Proof of Theorem 2.3. Let $\beta \geq 0$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $0 < \mu < \tilde{\mu} \leq 1$ and $\omega < 0$.

(a) Assume that $\alpha > 0$. Since A is ω -sectorial of angle $0 < \theta \leq \tilde{\mu}\frac{\pi}{2}$, it follows from Theorem 2.1 and Corollary 2.2 that A generates an (α, β, μ) -resolvent family $(S_{\alpha, \beta}^\mu(t))_{t \geq 0}$. Recall the function h given by

$$h(\lambda) := \frac{\lambda(\lambda + \beta)^\mu}{(\lambda + \beta)^\mu + \alpha}.$$

We have that for every $\lambda \in \mathbb{C}$ such that $h(\lambda) \in \omega + \Sigma_\theta$, the resolvent $\left(\frac{\lambda(\lambda + \beta)^\mu}{(\lambda + \beta)^\mu + \alpha} I - A \right)^{-1}$ exists and the Laplace transform of $S_{\alpha, \beta}^\mu(t)$ is given by

$$\hat{S}_{\alpha, \beta}^\mu(\lambda) = \frac{(\lambda + \beta)^\mu}{(\lambda + \beta)^\mu + \alpha} \left(\frac{\lambda(\lambda + \beta)^\mu}{(\lambda + \beta)^\mu + \alpha} I - A \right)^{-1}. \quad (4.4)$$

By the uniqueness of the Laplace transform, it follows from (4.4) that

$$S_{\alpha, \beta}^\mu(t) = e^{-\beta t} G_{\alpha, \beta}^\mu(t) \quad \text{for all } t \geq 0, \quad (4.5)$$

where the Laplace transform of $G_{\alpha, \beta}^\mu(t)$ is given, for all $\lambda \in \mathbb{C}$ such that $h(\lambda - \beta) \in \omega + \Sigma_\theta$, by

$$\hat{G}_{\alpha, \beta}^\mu(\lambda) = \frac{\lambda^\mu}{\lambda^\mu + \alpha} \left(\frac{(\lambda - \beta)\lambda^\mu}{\lambda^\mu + \alpha} I - A \right)^{-1}.$$

Let

$$h(\lambda - \beta) = \tilde{h}(\lambda) = \frac{\lambda^\mu(\lambda - \beta)}{\lambda^\mu + \alpha}.$$

There exists a constant $M > 0$ such that for all $\lambda \in \mathbb{C}$ with $\tilde{h}(\lambda) \in \omega + \Sigma_\theta$, we have the estimate

$$\begin{aligned} \left\| \left(\frac{(\lambda - \beta)\lambda^\mu}{\lambda^\mu + \alpha} I - A \right)^{-1} \right\| &\leq \frac{M}{\left| \frac{(\lambda - \beta)\lambda^\mu}{\lambda^\mu + \alpha} - \omega \right|} \\ &= \frac{M|\lambda^\mu + \alpha|}{|\lambda^{\mu+1} - (\beta + \omega)\lambda^\mu - \alpha\omega|}. \end{aligned} \quad (4.6)$$

Note that $\alpha\omega < 0$ and by hypothesis $-(\omega + \beta) \geq 0$. Let

$$g(t) = t^{\mu+1}, \quad t > 0.$$

We exploit some ideas from the proof of [4, Theorem 1]. Recall that A is ω -sectorial of angle $\theta = \tilde{\mu}\frac{\pi}{2}$. Using the inversion formula for the Laplace transform, we have that

$$G_{\alpha,\beta}^\mu(t) = \frac{1}{2\pi i} \int_\gamma e^{\lambda t} \frac{\lambda^\mu}{\lambda^\mu + \alpha} \left(\frac{(\lambda - \beta)\lambda^\mu}{\lambda^\mu + \alpha} I - A \right)^{-1} d\lambda, \quad (4.7)$$

where γ is a positively oriented path whose support Γ is given by

$$\Gamma := \{\lambda : \lambda \in \mathbb{C}, \quad \lambda^{\mu+1} \text{ belongs to the boundary of } B_{\frac{1}{g(t)}}, \quad t > 0\},$$

where for $\delta > 0$,

$$B_\delta := \{\delta + \Sigma_\theta\} \cup S_\phi,$$

and

$$S_\phi := \{\lambda \in \mathbb{C} : |\arg(\lambda)| < \phi\}, \quad (\mu + 1)\frac{\pi}{2} < \phi < \frac{\pi}{2} + \theta.$$

It follows from Lemma 4.1 that for such λ , we have that $\tilde{h}(\lambda) \in \omega + \Sigma_\theta$. Hence, the resolvent $\left(\frac{(\lambda - \beta)\lambda^\mu}{\lambda^\mu + \alpha} I - A \right)^{-1}$ is well-defined. Thus the representation (4.7) of $G_{\alpha,\beta}^\mu(t)$ is meaningful.

Next, we split the path γ into two parts γ_1 and γ_2 whose supports Γ_1 and Γ_2 are the sets formed by the complex numbers λ such that $\lambda^{\mu+1}$ lies on the intersection of Γ and the boundaries of $\frac{1}{g(t)} + \Sigma_\theta$ and S_ϕ respectively, that is,

$$\Gamma_1 = \Gamma \cap \overline{\left\{ \frac{1}{g(t)} + \Sigma_\theta \right\}} \quad \text{and} \quad \Gamma_2 = \Gamma \cap \overline{\{S_\phi\}}.$$

Thus $\Gamma = \Gamma_1 \cup \Gamma_2$ and $G_{\alpha,\beta}^\mu(t) = I_1(t) + I_2(t)$, for $t > 0$, where

$$I_j(t) := \frac{1}{2\pi i} \int_{\gamma_j} e^{\lambda t} \frac{\lambda^\mu}{\lambda^\mu + \alpha} \left(\frac{(\lambda - \beta)\lambda^\mu}{\lambda^\mu + \alpha} I - A \right)^{-1} d\lambda, \quad j = 1, 2.$$

Now, we estimate the norms $\|I_1(t)\|$ and $\|I_2(t)\|$.

Using (4.6) and (4.7) we get that for every $t > 0$,

$$\|I_1(t)\| \leq \frac{1}{2\pi} \int_{\gamma_1} |e^{\lambda t}| \left| \frac{\lambda^\mu}{\lambda^{\mu+1} - (\beta + \omega)\lambda^\mu - \alpha\omega} \right| |d\lambda|. \quad (4.8)$$

Let λ_{\min} be the complex $\lambda \in \mathbb{C}$ such that $\text{Im}(\lambda) > 0$, and $|\lambda^{\mu+1} - (\beta + \omega)\lambda^\mu - \alpha\omega| = \text{dist}(L, \alpha\omega)$, where L is the line passing through the point $(\frac{1}{g(t)}, 0)$ and the intersection of Γ_1 and Γ_2 . Then for $\lambda \in \Gamma_1$ we have that

$$\frac{1}{|\lambda^{\mu+1} - (\beta + \omega)\lambda^\mu - \alpha\omega|} \leq \frac{g(t)}{\cos(\theta)(1 + \alpha|\omega|g(t))}, \quad t > 0. \quad (4.9)$$

It follows from (4.8) and (4.9) that for every $t > 0$,

$$\|I_1(t)\| \leq \frac{M}{2\pi \cos(\theta)(1 + \alpha|\omega|g(t))} \int_{\gamma_1} |e^{\lambda t}| |\lambda|^\mu |d\lambda|. \quad (4.10)$$

Recall that

$$|\arg(\lambda^{\mu+1})| < \phi < \frac{\pi}{2} + \theta \leq (1 + \tilde{\mu})\frac{\pi}{2}.$$

Therefore the contour can be chosen such that

$$(\mu + 1)\frac{\pi}{2} < |\arg(\lambda^{\mu+1})| < \phi < \frac{\pi}{2} + \theta \leq (1 + \tilde{\mu})\frac{\pi}{2}.$$

Thus letting $\varphi = |\arg(\lambda)|$ we have that $\frac{\pi}{2} < \varphi < \frac{1+\tilde{\mu}}{1+\mu}\frac{\pi}{2}$. It follows from (4.10) that (note that $\cos(\varphi) < 0$) for every $t > 0$ we have that

$$\|I_1(t)\| \leq \frac{M}{2\pi \cos(\theta)(1 + \alpha|\omega|g(t))} \int_0^\infty e^{s \cos(\varphi)t} s^\mu ds \leq \frac{C}{1 + \alpha|\omega|g(t)}. \quad (4.11)$$

Similarly, if $\lambda \in \Gamma_2$, then for every $t > 0$, we have that

$$\frac{1}{|\lambda^{\mu+1} - (\beta + \omega)\lambda^\mu - \alpha\omega|} \leq \frac{1}{|\lambda^{\mu+1} - \alpha\omega|} \leq \frac{g(t)}{|\cos(\varphi)|}.$$

Hence there exists a constant $C > 0$ such that for every $t > 0$,

$$\begin{aligned} \|I_2(t)\| &\leq \frac{M}{2\pi |\cos(\varphi)|} \int_{\gamma_2} |e^{\lambda t}| |\lambda|^\mu |d\lambda| \\ &\leq Cg(t) \int_0^\infty e^{s \cos(\varphi)t} s^\mu ds = C. \end{aligned} \quad (4.12)$$

It follows from (4.11) and (4.12) that there exists a constant $C > 0$ such that

$$\|G_{\alpha,\beta}^\mu(t)\| \leq C \left[1 + \frac{1}{1 + \alpha|\omega|g(t)} \right], \quad \forall t > 0. \quad (4.13)$$

The estimate (2.1) follows from (4.5), (4.13) and the strong continuity of the resolvent family on $[0, \infty)$. The proof of part (a) is complete.

(b) Now assume that $\alpha < 0$ and $\alpha + \beta^\mu \geq |\alpha|$. We proceed similarly as in part (a) by considering again the same function $g(t) = t^{\mu+1}$, $t > 0$. But here we set

$$\Gamma_1 = \Gamma \cap \overline{\left\{ \alpha\omega + \frac{1}{g(t)} + \Sigma_\theta \right\}} \quad \text{and} \quad \Gamma_2 = \Gamma \cap \overline{\{S_\phi\}}.$$

Recall that here $\alpha\omega > 0$. The hypothesis implies that the representation (4.7) of $G_{\alpha,\beta}^\mu(t)$ is also valid in this case. On Γ_1 , we have that

$$\left| \frac{1}{\lambda^{\mu+1} - (\omega + \beta)\lambda^\mu - \alpha\omega} \right| \leq \frac{1}{|\lambda^{\mu+1} - \alpha\omega|} \leq \frac{g(t)}{\cos(\theta)}, \quad t > 0, \quad (4.14)$$

and there exists a constant $C > 0$ such that

$$|\lambda|^{\mu+1} \leq C \left(\alpha\omega + \frac{1}{g(t)} \right). \quad (4.15)$$

It follows from (4.15) that on Γ_1 , we have that for every $t > 0$,

$$|e^{t\lambda}| \leq e^{t|\lambda|} \leq e^{Ct \left(\alpha\omega + \frac{1}{t^{\mu+1}} \right)^{\frac{1}{\mu+1}}} \leq C e^{(\alpha\omega)^{\frac{1}{\mu+1}} t}. \quad (4.16)$$

Using (4.14), (4.15) and (4.16) we get that for every $t > 0$,

$$\|I_1(t)\| \leq \frac{C}{2\pi \cos(\theta)} \left(\alpha\omega + \frac{1}{g(t)} \right) e^{(\alpha\omega)^{\frac{1}{\mu+1}} t} \int_{\gamma_1} \frac{1}{|\lambda|} |d\lambda|. \quad (4.17)$$

Since

$$\text{length}(\gamma_1) \leq C \left(\alpha\omega + \frac{1}{g(t)} \right)^{\frac{1}{\mu+1}}, \quad t > 0$$

and on Γ_1 (by using the law of sines),

$$|\lambda| \geq \left[\cos(\theta) \left(\alpha\omega + \frac{1}{g(t)} \right) \right]^{\frac{1}{\mu+1}}, \quad t > 0,$$

it follows from (4.17) that for every $t > 0$,

$$\|I_1(t)\| \leq C \left[1 + \alpha\omega g(t) \right] e^{(\alpha\omega)^{\frac{1}{\mu+1}} t}. \quad (4.18)$$

Next, we consider the integral $I_2(t)$. Let z_t and \bar{z}_t be the intersection points of the boundary of $\alpha\omega + \frac{1}{g(t)} + \Sigma_\theta$ and S_ϕ , for which we have

$$|\lambda^{\mu+1} - (\omega + \beta)\lambda^\mu - \alpha\omega| \geq |\lambda^{\mu+1} - \alpha\omega| \geq |\cos(\phi)| |z_t|, \quad \lambda \in \Gamma_2,$$

and

$$|z_t| \geq C \left(\alpha\omega + \frac{1}{g(t)} \right), \quad t > 0.$$

The same bounds are also valid for the conjugate \bar{z}_t of z_t . Letting $\gamma_2(s) = se^{i\varphi}$, $s > 0$ (recall that $\cos(\varphi) < 0$), we get that for every $t > 0$,

$$\begin{aligned} \|I_2(t)\| &\leq \frac{1}{2\pi} \int_{\gamma_2} |e^{\lambda t}| \left| \frac{\lambda^\mu}{\lambda^{\mu+1} - (\omega + \beta)\lambda^\mu - \alpha\omega} \right| |d\lambda| \\ &\leq \frac{1}{2\pi \sin(\phi) |z_t|} \int_{\gamma_2} |e^{\lambda t}| |\lambda^\mu| |d\lambda| \\ &\leq \frac{C t^{\mu+1}}{1 + \alpha\omega g(t)} \int_0^\infty e^{t \cos(\varphi) s} s^\mu ds \\ &\leq \frac{C}{1 + \alpha\omega g(t)}. \end{aligned} \quad (4.19)$$

It follows from (4.18) and (4.19) that for every $t > 0$,

$$\|G_{\alpha,\beta}^\mu(t)\| \leq C [1 + \alpha\omega g(t)] e^{(\alpha\omega)^{\frac{1}{\mu+1}} t}. \quad (4.20)$$

Now the estimate (2.2) follows from (4.5), (4.20) and the strong continuity of the resolvent family on $[0, \infty)$. The proof is finished. \square

5. EXPLICIT REPRESENTATION OF RESOLVENT FAMILIES

In this section, we consider particular examples of the operator A and give a more explicit (than the one in (1.5)) representation of the resolvent family associated with the system (1.1) in terms of some special functions and we also investigate their precise exponential bound.

5.1. **The case where $A = \rho I$.** Here, we assume that the operator A is given by $A = \rho I$ for some $\rho \in \mathbb{R}$. Hence, the system (1.1) becomes

$$\begin{cases} u'(t) = \rho u(t) + \frac{\rho\alpha}{\Gamma(\mu)} \int_0^t e^{-\beta(t-s)} (t-s)^{\mu-1} u(s) ds, & t > 0, \\ u(0) = u_0. \end{cases} \quad (5.1)$$

We have the following explicit representation of solutions.

Proposition 5.1. *Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $0 < \mu \leq 1$, $\beta \geq 0$ and $A = \rho I$ for some $\rho \in \mathbb{R}$. Assume that $\alpha > 0$ or $\alpha < 0$ and $\alpha + \beta^\mu \geq |\alpha|$. Then the strongly continuous exponentially bounded resolvent family $S_{\alpha,\beta}^\mu$ associated with the system (5.1) is given by*

$$S_{\alpha,\beta}^\mu(t) = e^{-\beta t} \sum_{k=0}^{\infty} (\rho + \beta)^k t^k E_{\mu+1,k+1}^{(k+1)}(\alpha \rho t^{\mu+1}), \quad t \geq 0, \quad (5.2)$$

provided that the series converges and where

$$E_{\mu+1,k+1}^{(k+1)}(z) := \sum_{n=0}^{\infty} \frac{(k+n)! z^n}{n! k! \Gamma(n\mu + k + n + 1)}, \quad z \in \mathbb{C}, \quad (5.3)$$

denotes the generalized Mittag-Leffler function.

Proof. Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $0 < \mu \leq 1$, $\beta \geq 0$ and $A = \rho I$ for some $\rho \in \mathbb{R}$. Assume that $\alpha > 0$ or $\alpha < 0$ and $\alpha + \beta^\mu \geq |\alpha|$. Then it follows from Corollary 2.2 that there exists a strongly continuous resolvent family $S_{\alpha,\beta}^\mu$ of type $(M, \tilde{\omega})$ (for some $\tilde{\omega} > 0$) such that the unique solution of (5.1) is given by (1.5). In addition we have that the Laplace transform of $S_{\alpha,\beta}^\mu(t)$ is given by

$$\widehat{S}_{\alpha,\beta}^\mu(\lambda) := \frac{(\lambda + \beta)^\mu}{(\lambda + \beta)^{\mu+1} - (\rho + \beta)(\lambda + \beta)^\mu - \alpha\rho}, \quad \operatorname{Re}(\lambda) > \tilde{\omega}. \quad (5.4)$$

Using the properties of Laplace transform, we have that

$$S_{\alpha,\beta}^\mu(t) = e^{-\beta t} G_{\alpha,\beta}^\mu(t), \quad (5.5)$$

for some function $G_{\alpha,\beta}^\mu(t)$ whose Laplace transform is given for all $\operatorname{Re}(\lambda) > \tilde{\omega} + \beta$ by

$$\widehat{G}_{\alpha,\beta}^\mu(\lambda) = \frac{\lambda^\mu}{\lambda^{\mu+1} - (\rho + \beta)\lambda^\mu - \alpha\rho}. \quad (5.6)$$

Since

$$\left| \frac{(\rho + \beta)\lambda^\mu}{\lambda^{\mu+1} - \alpha\rho} \right| < 1,$$

for λ large enough, we have that

$$\begin{aligned} \widehat{G}_{\alpha,\beta}^\mu(\lambda) &= \frac{\lambda^\mu}{\lambda^{\mu+1} - (\rho + \beta)\lambda^\mu - \alpha\rho} = \frac{\lambda^\mu}{[\lambda^{\mu+1} - \alpha\rho][1 - \frac{(\rho + \beta)\lambda^\mu}{\lambda^{\mu+1} - \alpha\rho}]} \\ &= \frac{\lambda^\mu}{\lambda^{\mu+1} - \alpha\rho} \sum_{k=0}^{\infty} \frac{(\rho + \beta)^k \lambda^{k\mu}}{[\lambda^{\mu+1} - \alpha\rho]^k} = \sum_{k=0}^{\infty} \frac{(\rho + \beta)^k \lambda^{(k+1)\mu}}{[\lambda^{\mu+1} - \alpha\rho]^{k+1}} \\ &= \sum_{k=0}^{\infty} \frac{(\rho + \beta)^k \lambda^{2(k+1)\mu}}{\lambda^{(k+1)\mu} [\lambda^{\mu+1} - \alpha\rho]^{k+1}}. \end{aligned}$$

Taking the inverse Laplace transform, we get that (see e.g. [7, Formula 17.6])

$$\mathcal{L}^{-1} \left(\frac{\lambda^{2(k+1)\mu}}{\lambda^{(k+1)\mu} [\lambda^{\mu+1} - \alpha\rho]^{k+1}} \right) (t) = t^k E_{\mu+1, k+1}^{(k+1)}(\alpha\rho t^{\mu+1}),$$

where $E_{\mu+1, k+1}^{(k+1)}$ is the generalized Mittag-Leffler function given in (5.3). We have shown that

$$G_{\alpha, \beta}^{\mu}(t) = \sum_{k=0}^{\infty} (\rho + \beta)^k t^k E_{\mu+1, k+1}^{(k+1)}(\alpha\rho t^{\mu+1}). \quad (5.7)$$

Now (5.2) follows from (5.5) and (5.7). The proof is finished. \square

We have the following result as a direct consequence of Theorem 2.3.

Corollary 5.2. *Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $0 < \mu \leq 1$, $\beta \geq 0$, $A = \rho I$ for some $\rho < 0$ and assume in addition that $\rho + \beta \leq 0$. Then the following assertions hold.*

(a) *If $\alpha > 0$, then there exists a constant $M > 0$ such that*

$$\|S_{\alpha, \beta}^{\mu}(t)\| \leq M e^{-\beta t}, \quad \forall t \geq 0. \quad (5.8)$$

(b) *If $\alpha < 0$ and $\alpha + \beta^{\mu} \geq |\alpha|$, then there exists a constant $M > 0$ such that*

$$\|S_{\alpha, \beta}^{\mu}(t)\| \leq M (1 + \alpha\rho t^{\mu+1}) e^{-(\beta - (\alpha\rho)^{\frac{1}{\mu+1}})t}, \quad \forall t \geq 0. \quad (5.9)$$

Proof. This is a particular case of Theorem 2.3 since A is ω -sectorial of angle θ for every $0 < \theta < \frac{\pi}{2}$, with $\omega = \rho < 0$ and $\omega + \beta = \rho + \beta \leq 0$ by assumption. \square

In the following remark we show that if $\mu = 1$ one recovers some of the results contained in the references [1, 13].

Remark 5.3. *Let $\mu = 1$ and $A = \rho I$ for some $\rho < 0$. It follows from (5.6) that*

$$\widehat{G}_{\alpha, \beta}^1(\lambda) = \frac{\lambda}{\lambda^2 - (\rho + \beta)\lambda - \alpha\rho}.$$

Let

$$D := (\rho + \beta)^2 + 4\alpha\rho.$$

We have the following three situations.

(i) *If $D > 0$, then let $\lambda_1 := \frac{\rho + \beta + \sqrt{D}}{2}$ and $\lambda_2 := \frac{\rho + \beta - \sqrt{D}}{2}$. Using partial fractions, we get that*

$$\widehat{G}_{\alpha, \beta}^1(\lambda) = \frac{1}{\sqrt{D}} \left(\frac{\lambda_1}{\lambda - \lambda_1} - \frac{\lambda_2}{\lambda - \lambda_2} \right).$$

This implies that

$$G_{\alpha, \beta}^1(t) = \frac{\rho + \beta + \sqrt{D}}{2\sqrt{D}} e^{\frac{\rho + \beta + \sqrt{D}}{2}t} - \frac{\rho + \beta - \sqrt{D}}{2\sqrt{D}} e^{\frac{\rho + \beta - \sqrt{D}}{2}t}, \quad \forall t \geq 0,$$

so that

$$S_{\alpha, \beta}^1(t) = \frac{\rho + \beta + \sqrt{D}}{2\sqrt{D}} e^{\frac{\rho - \beta + \sqrt{D}}{2}t} - \frac{\rho + \beta - \sqrt{D}}{2\sqrt{D}} e^{\frac{\rho - \beta - \sqrt{D}}{2}t}, \quad \forall t \geq 0.$$

In that case one has uniform exponential stability if and only if

$$\rho - \beta + \sqrt{(\rho + \beta)^2 + 4\alpha\rho} < 0.$$

(ii) If $D = 0$, then

$$\widehat{G}_{\alpha,\beta}^1(\lambda) = \frac{\lambda}{\left(\lambda - \frac{\rho+\beta}{2}\right)^2} = \frac{1}{\lambda - \frac{\rho+\beta}{2}} + \frac{\rho+\beta}{2} \frac{1}{\left(\lambda - \frac{\rho+\beta}{2}\right)^2}.$$

This implies that

$$G_{\alpha,\beta}^1(t) = \left(1 + \frac{\rho+\beta}{2}t\right) e^{\frac{\rho+\beta}{2}t}, \quad \forall t \geq 0,$$

so that

$$S_{\alpha,\beta}^1(t) = \left(1 + \frac{\rho+\beta}{2}t\right) e^{\frac{\rho-\beta}{2}t}, \quad \forall t \geq 0.$$

In that case one has uniform exponential stability if and only if

$$\rho - \beta < 0.$$

(iii) Now if $D < 0$, then let $\lambda_0 := \frac{\rho+\beta+i\sqrt{-D}}{2}$ and $\overline{\lambda}_0 = \frac{\rho+\beta-i\sqrt{-D}}{2}$. Proceeding as above we get that for every $t \geq 0$,

$$S_{\alpha,\beta}^1(t) = \left(\cos(\sqrt{-D}t) - i\frac{\rho+\beta}{\sqrt{-D}}\sin(\sqrt{-D}t)\right) e^{\frac{\rho-\beta}{2}t},$$

so that one has uniform exponential stability if and only if

$$\rho - \beta < 0.$$

5.2. The case of a self-adjoint operator. We assume that $-A$ is a non-negative and self-adjoint operator with compact resolvent on the Hilbert space $L^2(\Omega)$ where $\Omega \subset \mathbb{R}^N$ is a bounded open set. Since $-A$ is non-negative and self-adjoint with compact resolvent, then it has a discrete spectrum which is formed of eigenvalues. Its eigenvalues satisfy $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots$ and $\lim_{n \rightarrow \infty} \lambda_n = \infty$. We denote the normalized eigenfunction associated with λ_n by ϕ_n . Then $\{\phi_n : n \in \mathbb{N}\}$ is an orthonormal basis for $L^2(\Omega)$ and the set is also total in $D(A)$. Observe also that for every $v \in D(A)$ we can write

$$-Av = \sum_{k=1}^{\infty} \lambda_k \langle v, \phi_k \rangle_{L^2(\Omega)} \phi_k.$$

We have the following result as a direct consequence of Proposition 5.1.

Corollary 5.4. Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $0 < \mu \leq 1$, $\beta > 0$ and let A be as above. Assume that $\alpha > 0$ or $\alpha < 0$ and $\alpha + \beta^\mu \geq |\alpha|$. Then the strongly continuous exponentially bounded resolvent family $S_{\alpha,\beta}^\mu$ associated with the system (1.1) is given by

$$S_{\alpha,\beta}^\mu(t) = e^{-\beta t} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} (\beta - \lambda_n)^k t^k E_{\mu+1,k+1}^{(k+1)}(-\alpha \lambda_n t^{\mu+1}), \quad t \geq 0, \quad (5.10)$$

provided that the series converges.

Proof. Let A be as above. Multiplying both sides of (1.1) by $\phi_n(x)$ and integrating over the set Ω we get that for every $n \in \mathbb{N}$, the function $u_n(t) := \langle u(t), \phi_n \rangle_{L^2(\Omega)}$ is a solution of the system

$$\begin{cases} u_n'(t) = -\lambda_n u_n(t) + \frac{-\lambda_n \alpha}{\Gamma(\mu)} \int_0^t e^{-\beta(t-s)} (t-s)^{\mu-1} u_n(s) ds, & t > 0 \\ u_n(0) = u_{0,n}, \end{cases}$$

where $u_{0,n} = \langle u_0, \phi_n \rangle_{L^2(\Omega)}$. It follows from Proposition 5.1 that

$$u_n(t) = S_{\alpha,\beta,n}^\mu(t) u_{0,n}, \quad \forall t \geq 0,$$

where for every $n \in \mathbb{N}$,

$$S_{\alpha,\beta,n}^\mu(t) = e^{-\beta t} \sum_{k=0}^{\infty} (\beta - \lambda_n)^k t^k E_{\mu+1,k+1}^{(k+1)}(-\alpha \lambda_n t^{\mu+1}), \quad \forall t \geq 0.$$

Since

$$u(t, x) = \sum_{n=1}^{\infty} u_n(t) \phi_n(x), \quad \forall t \geq 0, \quad x \in \Omega,$$

we have that

$$u(t, x) = \sum_{n=1}^{\infty} S_{\alpha,\beta,n}^\mu(t) u_{0,n} \phi_n(x), \quad \forall t \geq 0, \quad x \in \Omega,$$

so that $S_{\alpha,\beta}^\mu$ is given by the expression in (5.10). The proof is finished. \square

Corollary 5.5. *Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $0 < \mu \leq 1$, $\beta \geq 0$ and let A be as above. Assume that the first eigenvalue $\lambda_1 > 0$ and that $\beta - \lambda_1 \leq 0$. Then the following assertions hold.*

(a) *If $\alpha > 0$, then there exists a constant $M > 0$ such that*

$$\|S_{\alpha,\beta}^\mu(t)\| \leq M e^{-\beta t}, \quad \forall t \geq 0.$$

(b) *If $\alpha < 0$ and $\alpha + \beta^\mu \geq |\alpha|$, then there exists a constant $M > 0$ such that*

$$\|S_{\alpha,\beta}^\mu(t)\| \leq M (1 - \alpha \lambda_1 t^{\mu+1}) e^{-(\beta - (-\alpha \lambda_1)^{\frac{1}{\mu+1}})t}, \quad \forall t \geq 0.$$

Proof. This is a particular case of Theorem 2.3, given that by assumption, the operator A is ω -sectorial of angle θ for every $0 < \theta < \frac{\pi}{2}$, with $\omega = -\lambda_1 < 0$ and that $\beta + \omega = \beta - \lambda_1 \leq 0$. \square

REFERENCES

- [1] J. Chen, T. Xiao and J. Liang. *Uniform exponential stability of solutions to abstract Volterra equations*. J. Evol. Equ. **4** (9) (2009), 661-674.
- [2] J. Chen, J. Liang and T. Xiao. *Stability of solutions to integro-differential equations in Hilbert spaces*. Bull. Belg. Math. Soc. **18** (2011), 781-792.
- [3] C. Corduneanu. *Integral Equations and Applications*, Cambridge University Press, Cambridge, 1991.
- [4] E. Cuesta. *Asymptotic behaviour of the solutions of fractional integro-differential equations and some time discretizations*. Discrete Contin. Dyn. Syst. 2007, Dynamical Systems and Differential Equations. Proceedings of the 6th AIMS International Conference, suppl., 277-285.
- [5] R. Gorenflo and F. Mainardi. *On Mittag-Leffler-type functions in fractional evolution processes*. J. Comp. Appl. Math. **118** (2000), 283-299.
- [6] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, Springer, New York, 2000.
- [7] H. Haubold, A. Mathai and R. Saxena. *Mittag-Leffler Functions and Their Applications*. J. Appl. Math., Article ID 298628.
- [8] M. Haase. *The functional Calculus for Sectorial Operators*. Operator Theory: Advances and Applications, **169**. Birkhäuser Verlag, Basel, 2006.
- [9] B. Jacob and J. R. Partington. *A resolvent test for admissibility of Volterra observation operators*. J. Math. Anal. Appl. **332** (2007), 346-355.
- [10] V. Keyantuo, C. Lizama and M. Warma. *Asymptotic behavior of fractional-order semilinear evolution equations*. Differential Integral Equations **26** (2013), 757-780.
- [11] C. Lizama. *Regularized solutions for abstract Volterra equations*. J. Math. Anal. Appl. **243** (2000), 278-292.
- [12] C. Lizama and J. Sánchez. *On perturbation of K -regularized resolvent families*. Taiwanese J. Math. **7** (2003), 217-227.

- [13] C. Lizama and R. Ponce. *Bounded solutions to a class of semilinear integro-differential equations in Banach spaces*. Nonlinear Anal. **74** (2011), 3397-3406.
- [14] K. Miller and B. Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, New York, 1993.
- [15] J. Prüss. *Evolutionary Integral Equations and Applications*. Monographs Math. **87**, Birkhäuser Verlag, Basel, 1993.
- [16] V. Vergara and R. Zacher. *Lyapunov functions and convergence to steady state for differential equations of fractional order*. Math. Z. **259** (2008), 287–309.

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